Esercizio 17

Calcolare

$$I = \iint_T 3x\,dxdy$$

ove

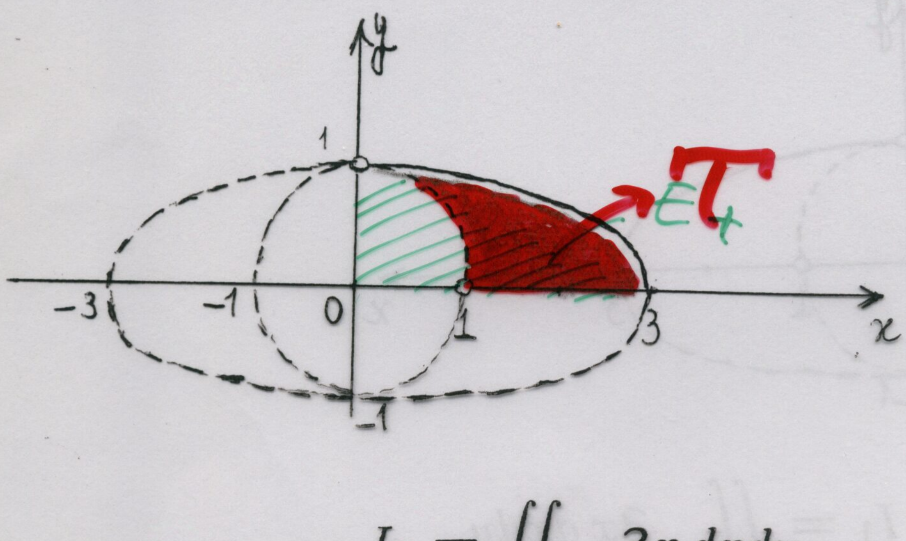
$$T = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1,\ x \geq 0,\ y \geq 0, \right.$$
$$\left. \frac{x^2}{9} + y^2 \leq 1\right\}$$

Osservo che

$$T = E_+ \setminus C_+$$

ove

$$E_+ = \left\{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0,\ \frac{x^2}{9} + y^2 \leq 1\right\}$$
$$C_+ = \left\{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\right\}$$



$$I_1 = \iint_{E_+} 3x\,dxdy$$
$$I_2 = \iint_{C_+} 3x\,dxdy$$

si ha che

$$\iint_T 3x\,dxdy = I_1 - I_2$$

<u>Calcolo I_1</u>: effettuo il cambiamento di coordinate (adatte alla geometria <u>ellittica</u> del problema)

$$\begin{cases} x = 3\rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases}$$

infatti $E_+ = \left\{ (x,y) \,/\, \frac{x^2}{9} + y^2 \leq 1,\ x \geq 0,\ y \geq 0 \right\}$

$$\Rightarrow\ \det J(\rho,\vartheta) = \det \begin{bmatrix} 3\cos(\vartheta) & -3\rho\sin(\vartheta) \\ \sin(\vartheta) & \rho\cos(\vartheta) \end{bmatrix} = 3\rho$$

$$E_+ \longrightarrow S_1 = \left\{ (\rho,\vartheta)\ : 0 \leq \rho \leq 1,\ 0 \leq \vartheta \leq \frac{\pi}{2} \right\}$$

$$= [0,1] \times \left[0, \frac{\pi}{2}\right]$$

Quindi

$$\iint_{E_+} 3x\,dxdy = 3\int_0^1 \left(\int_0^{\pi/2} 3\rho\cos(\vartheta) 3\rho\, d\vartheta \right) \cancel{dr}\ d\rho$$

$$= 27 \left(\int_0^1 \rho^2\, d\cancel{r}\rho \right) \cdot \left(\int_0^{\pi/2} \cos(\vartheta)\, d\vartheta \right)$$

$$= 27 \left[\frac{\rho^3}{3} \right]_0^1 \cdot [\sin(\vartheta)]_0^{\pi/2}$$

$$= 9$$

Calcolo I_2: cambiamento in coordinate polari (adatte alla simmetria circolare del dominio!)

$$\begin{cases} x = \rho\cos(\vartheta) \\ y = \rho\sin(\vartheta) \end{cases} \Rightarrow \det J(\rho, \vartheta) = \rho$$

$$C_+ \longrightarrow S_2 = \left\{(\rho, \vartheta) : 0 \le \rho \le 1,\ 0 \le \vartheta \le \frac{\pi}{2}\right\}$$
$$= [0,1] \times \left[0, \frac{\pi}{2}\right]$$

Quindi

$$\iint_{C_+} 3x\,dxdy = \int_0^1 \left(\int_0^{\pi/2} 3\rho\cos(\vartheta)\rho\,d\vartheta\right) d\rho$$
$$= \left(\int_0^1 3\rho^2\,d\rho\right) \cdot \left(\int_0^{\pi/2} \cos(\vartheta)\,d\vartheta\right)$$
$$= \left[\rho^3\right]_0^1 \cdot [\sin(\vartheta)]_0^{\pi/2} = 1$$

Concludo:

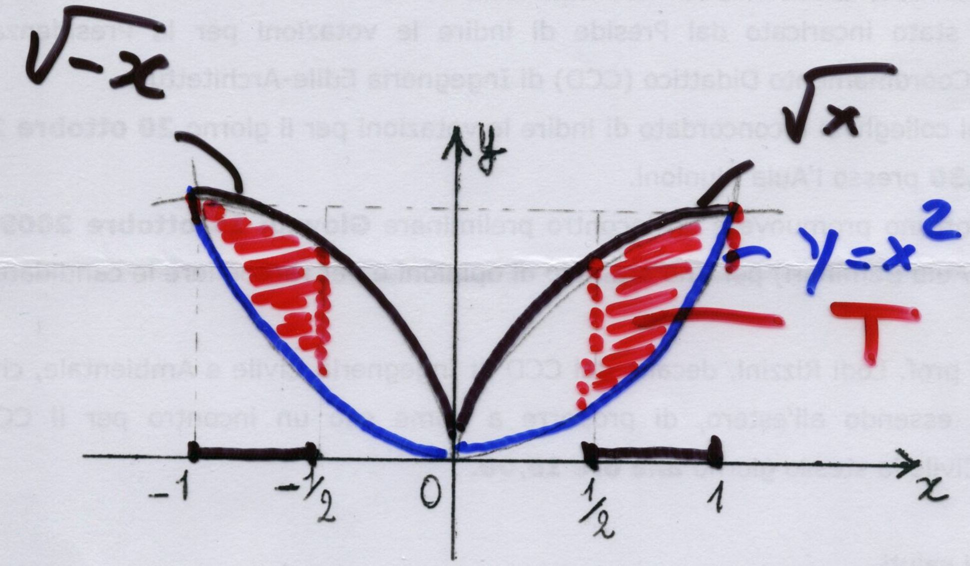
$$I = I_1 - I_2 = 9 - 1 = 8$$

Integrali doppi

Esercizio 12 (Scritto 10/7/06)

Calcolare $I = \iint_T \left(\frac{1}{3y} + 2xe^y \right) \, dxdy$, dove

$$T = \left\{ (x, y) \in \mathbb{R}^2 : \ \frac{1}{2} \le |x| \le 1, \ x^2 \le y \le \sqrt{|x|} \right\}$$



Abbiamo:

$$\frac{1}{2} \le |x| \le 1 \iff -1 \le x \le -\frac{1}{2} \text{ oppure } \frac{1}{2} \le x \le 1$$

Quando $-1 \le x \le -\frac{1}{2}$:

$$x^2 \le y \le \sqrt{|x|} \iff x^2 \le y \le \sqrt{-x}$$

Quando $\frac{1}{2} \le x \le 1$:

$$x^2 \le y \le \sqrt{|x|} \iff x^2 \le y \le \sqrt{x}$$

- f continua e limitata in T.

- <u>Simmetrie</u>:

T è simmetrico rispetto all'asse y

$f(x,y) = \frac{1}{3y} + 2xe^{-y} = f_1(x,y) + f_2(x,y)$. Abbiamo:

$$f_1(-x,y) = \frac{1}{y} = f_1(x,y)$$

$$f_2(-x,y) = 2(-x)e^{-y} = -f_2(x,y)\,.$$

Poniamo

$$T_+ = T \cap \{x \geq 0\}$$

$$= \left\{(x,y) \in \mathbb{R}^2 :\ \tfrac{1}{2} \leq x \leq 1\,,\ x^2 \leq y \leq \sqrt{x}\right\}.$$

Allora

$$I = \iint_T f(x,y)\,dxdy$$

$$= \iint_T f_1(x,y)\,dxdy + \iint_T f_2(x,y)\,dxdy$$

$$= 2\iint_{T_+} f_1(x,y)\,dxdy = \frac{2}{3}\iint_{T_+} \frac{1}{y}\,dxdy$$

Calcolo di $I_1 = \iint_{T_+} \frac{1}{y}\, dxdy$.

<u>Riduzione rispetto ad x:</u>

$$I_1 = \iint_{T_+} \frac{1}{y}\, dxdy = \int_{\frac{1}{2}}^{1} \left(\int_{x^2}^{\sqrt{x}} \frac{1}{y}\, dy \right) dx$$

$$= \int_{\frac{1}{2}}^{1} [\log(y)]_{x^2}^{\sqrt{x}}\, dx = \int_{\frac{1}{2}}^{1} (\log(\sqrt{x}) - \log(x^2))\, dx$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{2}\log(x) - 2\log(x) \right) dx$$

$$= -\frac{3}{2} \int_{\frac{1}{2}}^{1} \log(x)\, dx = -\frac{3}{2} \left\{ [x\log(x)]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} x \frac{1}{x}\, dx \right\}$$

$$= \frac{3}{4} \log\left(\frac{1}{2}\right) + \frac{3}{4} = \frac{3}{4}(1 - \log(2))$$

Quindi:

$$I = \frac{2}{3} I_1 = \frac{1}{2}(1 - \log(2))$$